Functional Relations for the R and T Operators on Plane-parallel Media*

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1. **Introduction** - In the classical theory of radiative transfer, one may associate with a separable plane-parallel medium, i.e., a medium in which the phase function $\rho = 4\pi\sigma/\alpha$ is independent of depth, two functions S and T, which analytically characterize the diffuse reflectance and transmittance of the medium. Explicit knowledge of the S and T functions is tantamount to a solution of the multiple scattering problem associated with the medium. Whereas the volume scattering function σ and the volume attenuation function α summarize the local optical properties of the medium, the S and T functions summarize the global optical properties of the medium. Furthermore, as the local behavior of the light field is described by the equation of transfer in terms of σ and α, so is the global behavior of the light field described by the principles of invariance in terms of S and T. The global approach was initiated by Ambarsumian, but it remained for Chandrasekhar to formulate the four principles of invariance and deduce from them the requisite integral equations for S and T for the case of finite, separable plane-parallel media. The importance of the global formulation has been illustrated by its role in many successful solutions of multiple scattering problems which hitherto were intractable under the local formulation.

In this note we consider the case of finite, non-separable plane-parallel media. As is shown below, the study of these media requires not two but four functions: a reflectance-transmittance pair for each of the two boundaries of the medium. The four principles of invariance for a non-separable plane-parallel medium are formulated, and are found to be just sufficient to yield the appropriate integro-differential equations governing the four functions.

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It is of some interest to observe that the following derivations can be patterned after the procedure used by Chandrasekhar for the separable case, namely the procedure which makes explicit use of the analytical statements of the principles of invariance in conjunction with the equation of transfer. This procedure promises to be adequate, under appropriate extensions, for use in the formulation of the functional relations governing the scattering functions in general one-parameter carrier spaces. In this way a useful methodology already extant in the classical theory is retained, and extended to more general settings. In addition, this procedure appears to be pertinant to an important question in the mathematical foundations of radiative transfer theory, namely the question of the apparent mathematical abiogenesis of the S and T functions and the principles of invariance. In short, are the principles of invariance part of the basic laws of radiative transfer theory or merely consequences of them? In this connection, we observe that the operator form of the equation of transfer adopted during the course of the present derivations, yields a pair of statements which clearly represent what might be termed the local forms of the principles of invariance. These in turn may possibly supply a key, by means of the general theory of Green's functions for linear operators, to an existance proof for the S and T functions and perhaps a derivation of the statements of the principles of invariance themselves. If this or any coterminous procedure is possible, then the general equation of transfer may serve as the sole starting point of an unbroken network of deductions covering all the salient mathematical features of modern radiative transfer theory.

The operators used in the present note are left in undecomposed form in order to secure maximum symmetry and compactness of the formulations.

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2. **Equation of Transfer and the Local Forms of the Principles of Invariance** - We begin with a plane parallel medium embedded in Euclidean

Figure 1

three space, namely that subset contained by two planes at geometric depths a and b from some datum plane. Several intermediate planes at depths x, y, z will also be considered: $a \leq x \leq y \leq z \leq b$. The volume attenuation and volume scattering functions, α, and σ, along with the radiance function N (specific intensity) will depend spatially only on the depth parameters, and in an arbitrary way. The steady state, emission free equation of transfer for radiance may be written

$$\mu \frac{dN(y,\mu,\phi)}{dy} = -\alpha(y)N(y,\mu,\phi) + N_*(y,\mu,\phi) , \qquad (1)$$

where

$$N_*(y,\mu,\phi) = \int_{\Xi} \sigma(y;\mu,\phi;\mu',\phi') N(y,\mu',\phi') d\mu' d\phi' .$$

We adopt the concepts of outward and inward radiances: $N(y,+\mu,\phi), N(y,-\mu,\phi), 0 < \mu \leq 1$

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and write the outward and inward radiance distributions at level y as $N_+(y), N_-(y)$;

these are <u>functions</u> which assign to the pair $(\mu, \phi), 0 < \mu \leq 1$, $0 < \phi \leq 2\pi$,

the appropriate radiance values at level y in the indicated directions. If Ξ_+

and Ξ_- are the totalities of outward and inward unit vectors at each level

y , the equation of transfer may be written in the following operator form:

$$- \frac{d N_+(y)}{dy} = N_+(y) \, t(y) + N_-(y) \, r(y) ,$$

$$\frac{d N_-(y)}{dy} = N_-(y) \, t(y) + N_+(y) \, r(y) , \qquad (2)$$

where for each y, $a \leq y \leq b$, and arbitrary (μ, ϕ), $0 < \mu \leq 1$, , $0 < \phi \leq 2\pi$,

$$r(y) = \frac{1}{\mu} \int_{\Xi_+} \sigma_-(y; \mu, \phi; \mu', \phi') [\ ] \, d\mu' d\phi' ,$$

$$t(y) = \frac{1}{\mu} \int_{\Xi_+} \sigma_+(y; \mu, \phi; \mu', \phi') [\ ] \, d\mu' d\phi' - \frac{1}{\mu} \alpha(y) .$$

Thus $r(y)$ and $t(y)$ evidentally play the role of local reflectance and

transmittance operators. σ_- and σ_+ are appropriate restrictions of σ chosen

so that their μ-arguments are positive. (Such restrictions are possible because

of the implicitly assumed isotropy of the medium). The pair (2) clearly represents

the <u>local</u> (or integro-differental) forms of the principles of invariance.

3. <u>Principles of Invariance for Non-separable Plane Parallel Media</u> - In an earlier

note[1] the classical principles of invariance were generalized to the context of

general one parameter carrier spaces. These generalizations were based directly

on considerations of the classical forms as given by Chandrasekhar, and resulted in

an explicit analytical embodiment of the principle of invariant imbedding[2] for the

radiative transfer context. The two main statements of the principles of invariance

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were shown to be of the forms:

$$
\begin{array}{ll}
\text{I} & N_+(y) = N_+(z)\,T(z,y) + N_-(y)\,R(y,z) \\
\text{II} & N_-(y) = N_-(x)\,T(x,y) + N_+(y)\,R(y,x)
\end{array}
\left.\vphantom{\begin{array}{l}1\\1\end{array}}\right\} \quad a \le x \le y \le z \le b.
\tag{3}
$$

The structural similarity of the sets (2), (3) can be noted. By suitably choosing the spread of the interval $[x, z]$ and the location of y within it, the operator forms of four principles of invariance for the present context may be obtained from (3):

I $\quad N_+(y) = N_+(b)\,T(b,y) + N_-(y)\,R(y,b)$.

II $\quad N_-(y) = N_-(a)\,T(a,y) + N_+(y)\,R(y,a)$.

III $\quad N_-(b) = N_-(a)\,T(a,b) + N_+(b)\,R(b,a) = N_-(y)\,T(y,b) + N_+(b)\,R(b,y)$.

IV $\quad N_+(a) = N_+(b)\,T(b,a) + N_-(a)\,R(a,b) = N_+(y)\,T(y,a) + N_-(a)\,R(a,y)$.

For every pair of depths (x, z) in the closed interval $[a, b]$ we have

$$
R(x,z) = \frac{1}{\mu} \int_{\Xi_+} R(x,z\,;\mu,\phi\,;\mu',\phi')\,[\quad]\,d\mu'd\phi'.
$$

Furthermore

$$
T(x,z) = T^\circ(x,z) + T^*(x,z),
$$

where

$$
T^*(x,z) = \frac{1}{\mu} \int_{\Xi_+} T(x,z\,;\mu,\phi\,;\mu',\phi')\,[\quad]\,d\mu'd\phi',
$$

$$
T^\circ(x,z) = \int_{\Xi_+} T_r(x)\,\delta(\mu-\mu')\,\delta(\phi-\phi')\,[\quad]\,d\mu'd\phi',
$$

in which

$$
T_r(x) = \exp\left\{-\frac{1}{\mu'}\left|\int_x^z \alpha(y)\,dy\right|\right\}, \quad r = |z-x|/\mu', \quad 0 < \mu' \le 1
$$

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$T(\mathbf{x}, \mathbf{z})$ is the transmittance operator; $T^{\circ}(x, z)$, $T^{*}(x, z)$ are its reduced and diffuse components. $R(x, z)$ is the reflectance operator, and since the reflected flux is by definition entirely of diffuse character, the corresponding reduced component $R^{\circ}(x, z)$ of $R(x, z)$ is the zero operator, so that $R^{*}(x, z) = R(x, z)$. It may be shown that, in analogy to the S and T functions of the classical theory, the present R and T functions have the properties:

$$\lim_{z \to x} R(x, z; \mu, \phi; \mu', \phi') / |z - x| = \sigma_{-}(x; \mu, \phi; \mu', \phi') ,$$

$$\lim_{z \to x} T(x, z; \mu, \phi; \mu', \phi') / |z - x| = \sigma_{+}(x; \mu, \phi; \mu', \phi') .$$

4. **Functional Relations for the R and T operators** - In what follows, let $N_{-}(a)$ be of arbitrary angular structure and set $N_{+}(b) = 0$ (the zero function), furthermore, all functions and operators will be assumed continuously differentiable with respect to the depth parameter y. Now from I, with the adopted boundary conditions in force,

$$\frac{dN_{+}(y)}{dy} = \frac{dN_{-}(y)}{dy} R(y, b) + N_{-}(y) \frac{dR(y, b)}{dy} . \tag{4}$$

Since interest is at present centered on the functional relations for the slab defined by the planes at depths a and b, we consider

$$\lim_{y \to a} \frac{dN_{+}(y)}{dy} ,$$

which, by (2), has the form:

$$- [N_{+}(a) t(a) + N_{-}(a) r(a)] = - N_{-}(a) [R(a, b) t(a) + r(a)] ,$$

where the latter relation follows form use of the left-hand equality of III. Similarly

$$\lim_{y \to a} \frac{dN_{-}(y)}{dy} = N_{-}(a) [t(a) + R(a, b) r(a)] .$$

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Since $\lim\limits_{y \to a} \dfrac{dR(y,b)}{dy} = \dfrac{\partial R(a,b)}{\partial a}$, and $N_-(a)$ is arbitrary, the limit operation $\lim\limits_{y \to a}$ applied to (4) yields:

I' $\qquad - \dfrac{\partial R(a,b)}{\partial a} = r(a) + t(a) R(a,b) + R(a,b) t(a) + R(a,b) r(a) R(a,b).$

Now from II,

$$\dfrac{dN_-(y)}{dy} = N_-(a) \dfrac{dT(a,y)}{dy} + \dfrac{dN_+(y)}{dy} R(y,a) + N_+(y) \dfrac{dR(y,a)}{dy} . \qquad (5)$$

By (2) and the left-hand equality of IV,

$$\lim\limits_{y \to b} \dfrac{dN_-(y)}{dy} = - N_-(a) T(a,b) t(b) ,$$

$$\lim\limits_{y \to b} \dfrac{dN_+(y)}{dy} = N_-(a) T(a,b) r(b) .$$

Applying the limit operation $\lim\limits_{y \to b}$ to (5), setting $\dfrac{\partial T(a,b)}{\partial b} = \lim\limits_{y \to b} \dfrac{dT(a,y)}{dy}$,

and recalling that $N_-(a)$ is arbitrary, we have

II' $\qquad \dfrac{\partial T(a,b)}{\partial b} = T(a,b) t(b) + T(a,b) r(b) R(b,a).$

The procedure is now clear. By applying this procedure in turn to the right-hand equalities of principles III, and IV, we have the results:

III' $\qquad \dfrac{\partial R(a,b)}{\partial b} = T(a,b) r(b) T(b,a) .$

IV' $\qquad - \dfrac{\partial T(a,b)}{\partial a} = t(a) T(a,b) + R(a,b) r(a) T(a,b).$

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In conclusion, several general observations may be made. (i) For non-separable spaces, i.e. spaces in which the phase function $\rho = 4\pi\sigma/\alpha$ is dependent on depth, I' - IV' show that the R and T operators possess <u>polarity</u>: $R(a,b) \neq R(b,a)$ $T(a,b) \neq T(b,a)$. (ii) the system I' - IV' is a simultaneous system of the simplest kind. It can be solved, in principle, by successively considering the individual relations in the order: I', IV', III', II'. (iii) To obtain the equations governing the R and T <u>functions</u>, one must first assume, as in the classical case, a Dirac-delta structure for $N_-(a)$: $N_-(a) = N^o \delta(\mu-\mu_o)\delta(\phi-\phi_o)$. Then by successively applying the operators on each side of the statements I' - **IV'** to $N_-(a)$ and making the appropriate reductions, the requisite integro-differential equations for the R and T functions are obtained.

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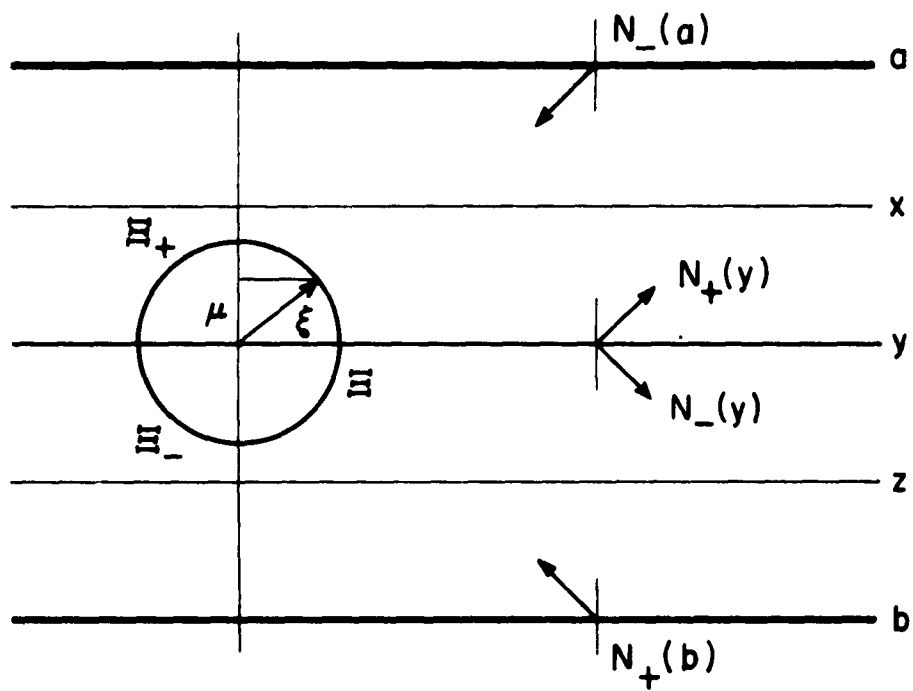
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Figure 1.